THE REGULARITY OF SOME VECTOR-VALUED VARIATIONAL INEQUALITIES WITH GRADIENT CONSTRAINTS

MOHAMMAD SAFDARI

Abstract. We prove the optimal regularity for some class of vector-valued variational inequalities with gradient constraints. We also give a new proof for the optimal regularity of some scalar variational inequalities with gradient constraints. In addition, we prove that some class of variational inequalities with gradient constraints are equivalent to an obstacle problem, both in the scalar and vector-valued case.

1. Introduction

Let $U \subset \mathbb{R}^n$ be an open bounded set. Suppose $K \subset \mathbb{R}^n$ is a balanced (symmetric with respect to the origin) compact convex set whose interior contains 0. Also suppose that $\boldsymbol{\eta} \in \mathbb{R}^N$ is a fixed nonzero vector. Consider the following problem of minimizing

$$I(\mathbf{v}) := \int_U |D\mathbf{v}|^2 - \boldsymbol{\eta} \cdot \mathbf{v}\, dx \tag{1.1}$$

over

$$K_1 := \{\mathbf{v} = (v^1, \cdots, v^N) \in H_0^1(U; \mathbb{R}^N) \mid \|D\mathbf{v}\|_{2,K} \le 1 \text{ a.e.}\}, \tag{1.2}$$

Where

$$\|A\|_{2,K} := \sup_{z \ne 0} \frac{|Az|}{\gamma_K(z)} \tag{1.3}$$

for an $N \times n$ matrix A, and γ_K is the norm associated to K defined by

$$\gamma_K(x) := \inf\{\lambda > 0 \mid x \in \lambda K\}. \tag{1.4}$$

As K_1 is a closed convex set and I is coercive, bounded and weakly sequentially lower semicontinuous, this problem has a unique solution $\mathbf{u}$. We will show that under some extra assumptions on K

$$\mathbf{u} \in C^{1,1}_{\text{loc}}(U; \mathbb{R}^N).$$

This problem is a generalization to the vector-valued case of the elastic-plastic torsion problem, which is the problem of minimizing

$$J_\eta(v) := \int_U |Dv|^2 - \eta v\, dx$$

for some $\eta > 0$, over

$$\{v \in H_0^1(U) \mid |Dv| \le 1 \text{ a.e.}\}.$$

Email address: safdari@berkeley.edu.

The regularity of the elastic-plastic torsion problem has been studied by Brezis and Stampacchia [2], and Caffarelli and Rivière [3]. There has been several extensions of their results to more general scalar problems with gradient constraints. See for example Jensen [8], Gerhardt [6], Evans [4], Wiegner [14], Ishii and Koike [7]. To the best of author's knowledge, the only work on the regularity of vector-valued problems with gradient constraints is Rozhkovskaya [12].

Our approach is to show that the above vector-valued problem is reducible to the scalar problem of minimizing J_1 over

$$\{v \in H_0^1(U) \mid |\boldsymbol{\eta}|Dv \in K^\circ \text{ a.e.}\},$$

where K° is the polar of K (See section 2). Then we show that this scalar problem is equivalent to a double obstacle problem with only Lipschitz obstacles. At the end, we generalize the proof of Caffarelli and Rivière [3], to obtain the optimal regularity. We should note that Lieberman [9] proves the regularity of a more general double obstacle problem by different methods.

In the process described above, we also show that our vector-valued problem with gradient constraint is equivalent to a vector-valued obstacle problem. This result, which is the first result of its kind as far as the author knows, is a generalization to the vector-valued case of the equivalence between the elastic-plastic torsion problem and an obstacle problem, proved by Brezis and Sibony [1]. Later Treu and Vornicescu [13] proved that the equivalence holds for a larger class of scalar variational inequalities with gradient constraints. We will further generalize their result. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are convex functions. Consider the problem of minimizing

$$J(v) := \int_U f(Dv(x)) + g(v(x))\, dx \tag{1.5}$$

over

$$W_K := \{v \in u_0 + W_0^{1,p}(U) \mid Dv(x) \in K \text{ a.e.}\}, \tag{1.6}$$

where $u_0 \in W^{1,p}(U)$. We will show that under appropriate assumptions, the minimizer of J over W_K is the same as its minimizer over

$$W_{u^-,u^+} := \{v \in u_0 + W_0^{1,p}(U) \mid u^-(x) \le v(x) \le u^+(x) \text{ a.e.}\}, \tag{1.7}$$

for some suitable functions u^-, u^+. The difference of our result with that of Treu and Vornicescu [13] is that we allow f, g to be only convex, and K to have empty interior. Some of our results has been proved using different means by Mariconda and Treu [10].

2. The Equivalence in the Scalar Case

Suppose $K \subset \mathbb{R}^n$ is a compact convex set whose interior contains the origin. Let J, W_K, and W_{u^-,u^+} be as above. We assume that on $W^{1,p}(U)$, J is finite, bounded below and sequentially weakly lower semicontinuous. These assumptions are satisfied if, for example, we impose some growth conditions on f, g and some mild regularity on ∂U. Therefore by our assumption, J attains its minimum on any nonempty closed convex subset of $W^{1,p}(U)$.

Furthermore, we assume that u_0 is Lipschitz, and

$$Du_0 \in K \qquad \text{a.e..}$$

Thus in particular, W_K is nonempty.

Definition 1. The **gauge** of K is a convex function defined by

$$\gamma_K(x) := \inf\{\lambda > 0 \mid x \in \lambda K\}, \tag{2.1}$$

and its **polar** is the convex set

$$K^\circ := \{x \mid x \cdot k \leq 1 \text{ for all } k \in K\}. \tag{2.2}$$

We recall that for all $x, y \in \mathbb{R}^n$, we have

$$x \cdot y \leq \gamma_K(x)\gamma_{K^\circ}(y). \tag{2.3}$$

Its proof can be found in Rockafellar [11]. Also, when K is balanced, K° is balanced too, and $\gamma_K, \gamma_{K^\circ}$ are both norms on $\mathbb{R}^n$.

Now, let us find $u^\pm \in W_K$ such that for all $u \in W_K$ we have $u^- \leq u \leq u^+$. Let $u^\pm$ be respectively the unique minimizers of $J^\pm(v) = \int_U \mp v(x)\, dx$ over W_K. We show that they have the desired property. We need the following lemma.

Lemma 1. *Suppose u is a compactly supported function in $W^{1,p}(\mathbb{R}^n)$ with $Du \in K$ a.e.. Then*

$$u(y) - u(x) \leq \gamma_{K^\circ}(y - x) \tag{2.4}$$

for all x, y.

Proof. Consider the mollifications

$$u_\epsilon(x) := (\eta_\epsilon \star u)(x) := \int_{B_\epsilon(x)} \eta_\epsilon(x - y)u(y)\, dy,$$

where η_ϵ is a nonnegative smooth function with support in $B_\epsilon(0)$, and $\int_{B_\epsilon(0)} \eta_\epsilon\, dx = 1$. Then we know that u_ϵ converges to u a.e., and $Du_\epsilon = \eta_\epsilon \star Du$. Hence

$$\begin{aligned}\gamma_K(Du_\epsilon(x)) &\leq \int_{B_\epsilon(x)} \gamma_K(\eta_\epsilon(x - y)Du(y))\, dy \\ &= \int_{B_\epsilon(x)} \eta_\epsilon(x - y)\gamma_K(Du(y))\, dy \ \leq 1,\end{aligned}$$

where we used Jensen's inequality in the first inequality. Thus

$$\begin{aligned}u_\epsilon(y) - u_\epsilon(x) &= \int_0^1 Du_\epsilon(x + t(y - x)) \cdot (y - x)\, dt \\ &\leq \int_0^1 \gamma_K(Du_\epsilon(x + t(y - x)))\gamma_{K^\circ}(y - x)\, dt \ \leq \gamma_{K^\circ}(y - x).\end{aligned}$$

Now we can let $\epsilon \to 0$ to obtain

$$u(y) - u(x) \leq \gamma_{K^\circ}(y - x) \qquad \text{for a.e. } x, y.$$

We can redefine u on the measure zero set where this relation fails, in a similar way that we extend Lipschitz functions to the closure of their domains. The extension will satisfy this relation everywhere. □

Lemma 2. *Each function in W_K is Lipschitz continuous. Also, W_K is bounded in $L^\infty(U)$ and in $W^{1,p}(U)$.*

Proof. To see this, let $u \in W_K$. Then $u = u_0 + v$ where $v \in W_0^{1,p}(U)$. Thus

$$|Dv| = |Du - Du_0| < 2R$$

for some $R > 0$. Now we can extend v by zero to all of $\mathbb{R}^n$, and the extension will satisfy the same gradient bound. Therefore by arguments similar to the previous lemma, we can see that the extension of v, and hence v itself, is Lipschitz with Lipschitz constant $2R$. Using the fact that v is zero on the boundary, this also implies that $\|v\|_{L^\infty} \leq 2RD$, where D is the diameter of U. The result for u follows easily, noting that u_0 is Lipschitz.

Now as $\|Du\|_{L^\infty} < C$ for some constant C independent of u, we have $\|Du\|_{L^p} < C$ since U is bounded. Noting that all $u \in W_K$ have the same boundary value, we get by Poincare inequality $\|u\|_{W^{1,p}} < C$. $\square$

Now we can see that $J^\pm$ are bounded on W_K. As $J^\pm$ are linear, they are weakly continuous. Furthermore W_K is convex, closed and bounded in $W^{1,p}(U)$. Hence W_K is compact with respect to sequential weak convergence. These imply that $J^\pm$ have minimizers over W_K. The uniqueness and the fact that $u^- \leq u^+$ a.e. on U, follows from a similar argument to the proof of the next lemma.

Lemma 3. *We have*

$$W_K \subset W_{u^-,u^+}.$$

Proof. Suppose $u \in W_K$, then $J^\pm(u^\pm) \leq J^\pm(u)$. Thus

$$\int_U \mp u^\pm \, dx \leq \int_U \mp u \, dx,$$

so

$$\int_U u^- \, dx \leq \int_U u \, dx \leq \int_U u^+ \, dx.$$

Suppose to the contrary that, for example, the set $E := \{x \mid u(x) > u^+(x)\}$ has positive measure. Consider the function

$$w(x) := \max(u, u^+) = \begin{cases} u^+(x) & x \notin E \\ u(x) & x \in E. \end{cases}$$

The derivative of w is

$$Dw(x) = \begin{cases} Du^+(x) & x \notin E \\ Du(x) & x \in E \end{cases} \qquad \text{for a.e. } x.$$

Therefore we have $Dw(x) \in K$ a.e.. Thus

$$J^+(w) = -\int_U w \, dx < -\int_U u^+ \, dx = J^+(u^+),$$

which is a contradiction. $\square$

The following characterization of $u^\pm$ will be used later. Here d_{K° is the metric associated to the norm γ_{K°.

Theorem 1. *Suppose u_0 equals a constant c everywhere. Then*

$$u^\pm(x) = c \pm d_{K^\circ}(x, \partial U).$$

Proof. It is enough to show that $c \pm d_{K^\circ}(x, \partial U)$ are the minimizers of $J^\pm$. The fact that $c \pm d_{K^\circ}(x, \partial U)$ belong to W_K is equivalent to the fact that $d_{K^\circ}(x, \partial U)$ is in $W_0^{1,p}(U)$ and its derivative has γ_K norm less than one. But $d_{K^\circ}(x, \partial U)$ is a Lipschitz function that vanishes on the boundary of U. It also satisfies

$$d_{K^\circ}(x, \partial U) - d_{K^\circ}(y, \partial U) \leq \gamma_{K^\circ}(x - y).$$

As proved by Treu and Vornicescu [13], this last property implies that the γ_K norm of the derivative of $d_{K^\circ}(x, \partial U)$ is less than or equal to 1 a.e..

Now similarly to the proof of Lemma 1, we can show that

$$|v(x) - c| \leq d_{K^\circ}(x, \partial U)$$

for all $v \in W_K$. Therefore $c \pm d_{K^\circ}(x, \partial U)$ minimize $J^\pm$ over W_K. □

The following theorem is the generalization of the result of Treu and Vornicescu [13]. We removed the assumptions on the derivatives of g, and allowed K to have empty interior.

Theorem 2. *Suppose K is a compact convex set containing 0, and u_0 is the restriction to U of a compactly supported function in $W^{1,p}(\mathbb{R}^n)$ with gradient a.e. in K. Also, suppose f, g are convex and at least one of them is strictly convex. Then the minimizer of*

$$J(v) = \int_U f(Dv(x)) + g(v(x))\, dx$$

over W_{u^-,u^+} is the same as its minimizer over W_K.

Proof. Note that the convexity assumptions on f, g imply that the minimizer of J over any nonempty convex closed set is unique. Also the assumption on u_0 implies $u_0(y) - u_0(x) \leq \gamma_{K^\circ}(y - x)$ for all $x, y \in U$, by Lemma 1. Let the minimizer of J over W_{u^-,u^+} be u. As $W_K \subset W_{u^-,u^+}$, it is enough to show that $u \in W_K$.

First assume that 0 is in the interior of K, and g is C^1 with strictly increasing derivative.

Similarly to Treu and Vornicescu [13], using u_0 we can extend $u^\pm$ and u to all of $\mathbb{R}^n$ in a way that the gradient of $u^\pm$ is still in K. Fix a nonzero vector $h \in \mathbb{R}^n$, and define

$$\begin{aligned} u_h^+(x) &:= \max\{u(x+h) - \gamma_{K^\circ}(h), u(x)\} \\ u_h^-(x) &:= \min\{u(x-h) + \gamma_{K^\circ}(h), u(x)\}, \end{aligned}$$

and

$$\begin{aligned} E^+ &:= \{x \in \mathbb{R}^n \mid u_h^+(x) = u(x+h) - \gamma_{K^\circ}(h) > u(x)\} \\ E^- &:= \{x \in \mathbb{R}^n \mid u_h^-(x) = u(x-h) + \gamma_{K^\circ}(h) < u(x)\}. \end{aligned}$$

The following assertions are easy to check

i) $u_h^\pm \in W_{u^-,u^+}$.

ii) $E^\pm \backslash U$ have measure zero.

iii) $E^+ = E^- - h$.

Now for any $0 < \lambda < 1$ we have $(i = 1, \cdots, m)$

$$\begin{aligned} J(u + \lambda(u_h^+ - u)) - J(u) = \int_{E^+} & f(Du(x) + \lambda(Du(x+h) - Du(x))) \\ & - f(Du(x)) + g(u(x) + \lambda(u(x+h) - \gamma_{K^\circ}(h) - u(x))) - g(u(x))\, dx \geq 0, \end{aligned} \tag{2.5}$$

and

$$J(u+\lambda(u_h^- - u)) - J(u) = \int_{E^-} f(Du(x)+\lambda(Du(x-h)-Du(x)))$$

$$- f(Du(x)) + g(u(x)+\lambda(u(x-h)+\gamma_{K^\circ}(h)-u(x))) - g(u(x))\,dx \geq 0. \tag{2.6}$$

By changing the variable from x to $x+h$ in the last integral, we get

$$\int_{E^+} f(Du(x+h)+\lambda(Du(x)-Du(x+h))) - f(Du(x+h))$$

$$+ g(u(x+h)+\lambda(u(x)+\gamma_{K^\circ}(h)-u(x+h))) - g(u(x+h))\,dx \geq 0. \tag{2.7}$$

Adding this to the first integral and using the convexity of f, we have

$$\int_{E^+} g(u(x+h)+\lambda(u(x)+\gamma_{K^\circ}(h)-u(x+h))) - g(u(x+h))$$

$$+ g(u(x)+\lambda(u(x+h)-\gamma_{K^\circ}(h)-u(x))) - g(u(x))\,dx \geq 0. \tag{2.8}$$

We divide this inequality by $\lambda > 0$ and take the limit as $\lambda \to 0$. Then, as g is C^1 and u is bounded, by Dominated Convergence Theorem we get

$$\int_{E^+} [g'(u(x+h)) - g'(u(x))](u(x)-u(x+h)+\gamma_{K^\circ}(h))\,dx \geq 0. \tag{2.9}$$

But on E^+, $u(x)-u(x+h)+\gamma_{K^\circ}(h) < 0$. Also g' is strictly increasing and therefore $g'(u(x+h))-g'(u(x)) > 0$. Hence E^+ must have measure zero. This means that for a.e. $x \in \mathbb{R}^n$

$$u(x+h)-u(x) \leq \gamma_{K^\circ}(h).$$

Taking $h \to 0$ (through a countable sequence) we get

$$D_h u(x) \leq \gamma_{K^\circ}(h).$$

Which implies $\gamma_K(Du(x)) \leq 1$, and this is equivalent to $u \in W_K$.

Now suppose that we only have $0 \in K$. Let

$$K_i := \{x+y \mid x \in K\,,\, |y| \leq \frac{1}{i}\} = \{z \mid d(z,K) \leq \frac{1}{i}\}.$$

Then $\{K_i\}$ is a decreasing family of compact convex sets containing K with $0 \in \operatorname{int} K_i$. Therefore $\{W_{K_i}\}$ is also a decreasing family containing W_K. Let $u_i^{\pm}$ be the corresponding obstacles to W_{K_i}. Then we have $u_i^+ \geq u^+$ and $u_i^- \leq u^-$. Also u_i^+ decreases with i, and u_i^- increases with i. Thus $\{W_{u_i^-,u_i^+}\}$ is a decreasing family too and contains W_{u^-,u^+}.

Let u_i be the minimizer of J over W_{K_i}. We have $Du_0 \in K \subset K_i$. Therefore we can apply the previous argument and we have $J(u_i) \leq J(v)$ for all $v \in W_{u_i^-,u_i^+} \supset W_{u^-,u^+}$. Now as u_i's are all in W_{K_1} we have $\|u_i\|_{W^{1,p}} < C$ for some universal C.

Therefore there is a subsequence of u_i's, where we denote it by u_{i_k}, which converges weakly in $u_0 + W_0^{1,p}(U)$ to u. By weak lower semicontinuity of J we get $J(u) \leq \liminf J(u_{i_k}) \leq J(v)$ for all $v \in W_{u^-,u^+}$. Thus to finish the proof we only need to show that $u \in W_K$. To see this note that the sequence u_{i_k} is eventually in each $W_{K_{i_k}}$ and as these are closed convex sets they are weakly closed, hence $u \in W_{K_{i_k}}$ for all k. This means $d(Du,K) \leq \frac{1}{i_k}$ a.e.. Thus $d(Du,K)=0$ a.e., and by closedness of K we get the desired result.

Next suppose that g is only convex. Consider the mollifications $g_\epsilon := \eta_\epsilon \star g$, where η_ϵ is the standard mollifier. First let us show that g_ϵ is convex too. We have

$$\begin{aligned} g_\epsilon(\lambda x + (1-\lambda)y) &= \int \eta_\epsilon(z) g(\lambda x + (1-\lambda)y - z)\, dz \\ &\le \int \eta_\epsilon(z)[\lambda g(x-z) + (1-\lambda)g(y-z)]\, dz \\ &\le \lambda g_\epsilon(x) + (1-\lambda)g_\epsilon(y). \end{aligned}$$

Now let

$$J_i(v) := \int_U f(Dv) + g_{\frac{1}{i}}(v) + \frac{1}{i}v^2\, dx.$$

Then since $g_\epsilon(v) + \epsilon v^2$ is a smooth strictly convex function, it has strictly increasing derivative. Let u_i be the minimizer of J_i over W_K. Then by the above we have $J_i(u_i) \le J_i(v)$ for all $v \in W_{u^-,u^+}$. As the u_i's are in W_K, and W_K is bounded in $W^{1,p}$, we can say that there is a subsequence of u_i, which we continue to denote it by u_i, that converges weakly to $u \in W_K$.

Since g_ϵ uniformly converges to g on compact sets, and for $v \in W_{u^-,u^+}$ we have $\|v\|_{L^\infty} < C$ for some constant C independent of v, we have for ϵ small enough and independent of v

$$|J_i(v) - J(v)| \le \int_U |g_{\frac{1}{i}}(v) - g(v)| + \frac{1}{i}v^2\, dx < \delta, \tag{2.10}$$

for i large enough. Hence $J(u_i) \le J(v) + 2\delta$. Then by weak lower semicontinuity of J we have $J(u) \le \liminf J(u_i) \le J(v) + 2\delta$. Since δ is arbitrary we get that u is the minimizer of J over W_{u^-,u^+} as required. □

Remark 1. We can also prove a version of this theorem when $0 \notin K$, by translating K. But we need to have a bound on the distance of K and the origin.

3. The Equivalence in The Vector-Valued Case

Suppose $K \subset \mathbb{R}^n$ is a balanced compact convex set whose interior contains 0. Also suppose that $\boldsymbol{\eta} \in \mathbb{R}^N$ is a fixed nonzero vector. Consider the following problems of minimizing

$$I(\mathbf{v}) := \int_U |D\mathbf{v}|^2 - \boldsymbol{\eta} \cdot \mathbf{v}\, dx \tag{3.1}$$

over

$$K_1 := \{\mathbf{v} = (v^1, \cdots, v^N) \in H_0^1(U; \mathbb{R}^N) \mid \|D\mathbf{v}\|_{2,K} \le 1 \text{ a.e.}\}, \tag{3.2}$$

and over

$$K_2 := \{\mathbf{v} = (v^1, \cdots, v^N) \in H_0^1(U; \mathbb{R}^N) \mid |\mathbf{v}(x)| \le d_K(x, \partial U) \text{ a.e.}\}. \tag{3.3}$$

Where

$$\|A\|_{2,K} := \sup_{z \neq 0} \frac{|Az|}{\gamma_K(z)} \tag{3.4}$$

for an $n \times n$ matrix A, and γ_K, d_K are respectively the norm associated to K and the metric of that norm. We show that these problems are equivalent.

As both K_1, K_2 are closed convex sets and I is coercive, bounded and weakly sequentially lower semicontinuous, both problems have unique solution.

Lemma 4. *We have*

$$K_1 \subseteq K_2.$$

Proof. To see this let $\mathbf{v} \in K_1$. Similarly to the proof of Lemma 1 we obtain

$$|\mathbf{v}(y) - \mathbf{v}(x)| \le \gamma_K(y - x) \tag{3.5}$$

for a.e. x, y. Using this relation we can redefine $\mathbf{v}$ on a set of measure zero the same way that we extend Lipschitz functions. Therefore we can assume that $\mathbf{v}$ is continuous. Now as $\mathbf{v}$ is 0 on ∂U, we can choose x to be the closest point on ∂U to y with respect to d_K, and get the desired result. $\square$

Lemma 5. *Let $\mathbf{u} = (u^1, \cdots, u^N)$ be the minimizer of I over K_2, and let*

$$T = (T^k_l) : \mathbb{R}^N \to \mathbb{R}^N$$

be an orthogonal linear map that fixes $\boldsymbol{\eta}$. Then $T\mathbf{u} \in K_2$ and

$$I(T\mathbf{u}) = I(\mathbf{u}). \tag{3.6}$$

Proof. To see this note that $T\mathbf{u} \in H^1_0(U; \mathbb{R}^N)$ and as T preserves the norm, for a.e. x we have

$$|T\mathbf{u}(x)| = |\mathbf{u}(x)| \le d_K(x, \partial U). \tag{3.7}$$

Furthermore as T is orthogonal we have

$$|DT\mathbf{u}|^2 = \sum_i \sum_k (T^k_l D_i u^l)^2 = \sum_i \sum_l (D_i u^l)^2 = |D\mathbf{u}|^2. \tag{3.8}$$

Hence (since $T\boldsymbol{\eta} = \boldsymbol{\eta}$ and T is orthogonal)

$$\begin{aligned} I(T\mathbf{u}) &= \int_U |DT\mathbf{u}|^2 - \boldsymbol{\eta} \cdot T\mathbf{u}\, dx \\ &= \int_U |D\mathbf{u}|^2 - T\boldsymbol{\eta} \cdot T\mathbf{u}\, dx \\ &= \int_U |D\mathbf{u}|^2 - \boldsymbol{\eta} \cdot \mathbf{u}\, dx = I(\mathbf{u}). \end{aligned}$$

$\square$

Theorem 3. *We have*

$$\mathbf{u}(x) = u(x)\boldsymbol{\eta}, \tag{3.9}$$

where u is the minimizer of

$$J_1(v) := \int_U |Dv|^2 - v\, dx \tag{3.10}$$

over

$$K_3 := \{v \in H^1_0(U; \mathbb{R}) \mid |v(x)| \le \frac{1}{|\boldsymbol{\eta}|} d_K(x, \partial U) \ \text{a.e.}\}. \tag{3.11}$$

Proof. By the above lemma and uniqueness of the minimizer, we must have $T\mathbf{u} = \mathbf{u}$ for all orthogonal linear maps T that fix $\boldsymbol{\eta}$. This implies that $\mathbf{u}(x) = u(x)\boldsymbol{\eta}$ for some scalar function u. Now we have

$$|u(x)\boldsymbol{\eta}| = |\mathbf{u}| \le d_K(x, \partial U).$$

Hence for a.e. x

$$|u(x)| \leq \frac{1}{|\boldsymbol{\eta}|} d_K(x, \partial U). \tag{3.12}$$

Also we have

$$D_i \mathbf{u} = D_i u \boldsymbol{\eta}.$$

Thus

$$I(\mathbf{u}) = \int_U |\boldsymbol{\eta}|^2 |Du|^2 - |\boldsymbol{\eta}|^2 u \, dx = |\boldsymbol{\eta}|^2 \int_U |Du|^2 - u \, dx = |\boldsymbol{\eta}|^2 J_1(u). \tag{3.13}$$

It is easy to see that u is the minimizer of J_1 over K_3. Because for any $w \in K_3$ we have $w\boldsymbol{\eta} \in K_2$, therefore

$$J_1(u) = |\boldsymbol{\eta}|^{-2} I(u\boldsymbol{\eta}) = |\boldsymbol{\eta}|^{-2} I(\mathbf{u}) \leq |\boldsymbol{\eta}|^{-2} I(w\boldsymbol{\eta}) = J_1(w).$$

□

Theorem 4. *The minimizer of I over K_2 is the same as its minimizer over K_1.*

Proof. By the above theorem

$$\mathbf{u}(x) = u(x)\boldsymbol{\eta},$$

where u is the minimizer of J_1 over K_3. But we know that the minimizer of J_1 over K_3 is the same as its minimizer over

$$K_4 := \{v \in H_0^1(U; \mathbb{R}) \mid \gamma_{K^\circ}(Dv) \leq \frac{1}{|\boldsymbol{\eta}|} \text{ a.e.}\}. \tag{3.14}$$

Therefore for all $z \in \mathbb{R}^n$, we have a.e.

$$\begin{aligned} |D\mathbf{u} \cdot z|^2 &= \sum_l \sum_i (D_i u^l z^i)^2 = \sum_l \sum_i (D_i u \eta^l z^i)^2 \\ &= \sum_i (D_i u z^i)^2 \sum_l (\eta^l)^2 = |\boldsymbol{\eta}|^2 |Du \cdot z|^2 \\ &\leq |\boldsymbol{\eta}|^2 \gamma_{K^\circ}(Du)^2 \gamma_K(z)^2 \leq \gamma_K(z)^2. \end{aligned}$$

This means that

$$\|D\mathbf{u}\|_{2,K} \leq 1 \qquad \text{a.e.}. \tag{3.15}$$

Hence $\mathbf{u} \in K_1$. Since $K_1 \subseteq K_2$, $\mathbf{u}$ is also the minimizer of I over K_1. □

4. The Optimal Regularity

Let

$$J_\eta(v) := \int_U \frac{1}{2} |Dv|^2 - \eta v \, dx.$$

Suppose $K \subset \mathbb{R}^n$ is a balanced compact convex set whose interior contains 0. Let u be the minimizer of J_η over

$$W_K := \{v \in c + H_0^1(U) \mid \gamma_K(Dv) \leq k \text{ a.e.}\},$$

where c, k are constants and γ_K is the gauge function of K. We showed that u is also the minimizer of J_η over

$$\{v \in c + H_0^1(U) \mid c - k d_{K^\circ}(x, \partial U) \leq v(x) \leq c + k d_{K^\circ}(x, \partial U) \text{ a.e.}\},$$

where K° is the polar of K, and d_{K° is the metric associated to the norm γ_{K°.

By the above assumptions, there is $A > 0$ such that $\gamma_{K^\circ}(x) \le A|x|$ for all x. We also need some sort of bound on the second derivative of γ_{K°, hence we assume that

$$\frac{\gamma_{K^\circ}(x+hz) + \gamma_{K^\circ}(x-hz) - 2\gamma_{K^\circ}(x)}{h^2} \le \frac{B}{\gamma_{K^\circ}(x) - h}, \tag{4.1}$$

where $B \ge 1$ is a constant, $\gamma_{K^\circ}(z) = 1$ and $h < \gamma_{K^\circ}(x)$.

Lemma 6. *The above inequality holds when γ_{K° is the p-norm for $p \ge 2$. (In this case, K is the unit disk in the $\frac{p}{p-1}$-norm.)*

Proof. Let $\gamma_p(x) = (\sum |x_i|^p)^{1/p}$ then for $\gamma_p(x) \ne 0$ we have

$$D_i\gamma_p(x) = |x_i|^{p-1}\mathrm{sgn}(x_i)(\sum |x_j|^p)^{1/p-1} = \frac{|x_i|^{p-1}\mathrm{sgn}(x_i)}{\gamma_p(x)^{p-1}}, \tag{4.2}$$

where $\mathrm{sgn}(x_i)$ is the sign of x_i. Thus

$$D^2_{ij}\gamma_p(x) = (p-1)|x_i|^{p-2}\delta_{ij}\frac{1}{\gamma_p(x)^{p-1}} - (p-1)|x_i|^{p-1}|x_j|^{p-1}\frac{\mathrm{sgn}(x_i)\mathrm{sgn}(x_j)}{\gamma_p(x)^{2p-1}}. \tag{4.3}$$

Hence

$$\begin{aligned} D^2_{zz}\gamma_p(x) &= \sum D^2_{ij}\gamma_p(x) z_i z_j \\ &= \frac{p-1}{\gamma_p(x)^{p-1}} \sum |x_i|^{p-2} z_i^2 - \frac{p-1}{\gamma_p(x)^{2p-1}} (\sum \mathrm{sgn}(x_i)|x_i|^{p-1} z_i)^2. \end{aligned}$$

By Holder's inequality we get

$$D^2_{zz}\gamma_p(x) \le \frac{p-1}{\gamma_p(x)^{p-1}} (\sum (|x_i|^{p-2})^{\frac{p}{p-2}})^{\frac{p-2}{p}} (\sum (z_i^2)^{\frac{p}{2}})^{\frac{2}{p}} = \frac{p-1}{\gamma_p(x)}\gamma_p(z)^2.$$

Thus if $\gamma_p(z) = 1$, we have

$$D^2_{zz}\gamma_p(x) \le \frac{p-1}{\gamma_p(x)}. \tag{4.4}$$

When $\gamma_p(x) > h$, γ_p is nonzero on the segment $L := \{x + \tau z \mid -h \le \tau \le h\}$; and so it is twice differentiable there. Therefore we can apply the mean value theorem to the restriction of γ_p and its first derivative to the segment L. Hence we get

$$\begin{aligned} \frac{\gamma_p(x+hz) + \gamma_p(x-hz) - 2\gamma_p(x)}{h^2} &= \frac{\gamma_p(x+hz) - \gamma_p(x) + \gamma_p(x-hz) - \gamma_p(x)}{h^2} \\ &= \frac{hD_z\gamma_p(x+sz) - hD_z\gamma_p(x-tz)}{h^2} \\ &= \frac{(s+t)}{h} D^2_{zz}\gamma_p(x+rz), \end{aligned}$$

where $0 < s, t < h$ and $-t < r < s$. Now as γ_p is convex, its second derivative is nonnegative definite. Hence

$$\begin{aligned} \frac{\gamma_p(x+hz) + \gamma_p(x-hz) - 2\gamma_p(x)}{h^2} &\le 2D^2_{zz}\gamma_p(x+rz) \\ &\le \frac{2(p-1)}{\gamma_p(x+rz)} \\ &\le \frac{2(p-1)}{\gamma_p(x) - h}. \end{aligned} \tag{4.5}$$

In the last inequality we used the triangle inequality for γ_p. $\square$

The following is our main regularity result. Note that by Theorem 3, we also get the regularity for the vector-valued case.

Theorem 5. *Suppose u is the minimizer of J_η over W_K. Then $u \in W^{2,\infty}_{loc}(U)$, and*

$$|D^2u(x)| \le C(n)\big[|\eta| + \frac{kA^2B}{d_{K^\circ}(x,\partial U)} + \frac{A^2|c|}{(d_{K^\circ}(x,\partial U))^2}\big], \tag{4.6}$$

where $C(n)$ is a constant depending only on the dimension n.

Proof. Let us assume that U has smooth boundary, we will remove this restriction at the end. We know that

$$\phi(x) = c - kd_{K^\circ}(x,\partial U) \le u(x) \le c + kd_{K^\circ}(x,\partial U) = \psi(x).$$

Let $\phi_\epsilon = \eta_\epsilon \star \phi + \delta_\epsilon$ and $\psi_\epsilon = \eta_\epsilon \star \psi$ where η_ϵ is the standard mollifier and $4kA\epsilon < \delta_\epsilon < 5kA\epsilon$ is chosen such that $\partial\{\phi_\epsilon < \psi_\epsilon\}$ is C^∞ (which is possible by Sard's Theorem). Note that

$$\begin{aligned} \{x \in U \mid d_{K^\circ}(x,\partial U) > 4A\epsilon\} \quad &\subset \{x \in \bar{U} \mid \phi_\epsilon(x) \le \psi_\epsilon(x)\} \\ &\subset \{x \in U \mid d_{K^\circ}(x,\partial U) > A\epsilon\}, \end{aligned} \tag{4.7}$$

as $\psi(x) - \phi(x) = 2kd_{K^\circ}(x,\partial U)$. Also

$$\begin{aligned} |\psi_\epsilon(x) - \psi(x)| \quad &\le \int_{|y|\le\epsilon} \eta_\epsilon(y)|\psi(x-y) - \psi(x)|\,dy \\ &\le \int_{|y|\le\epsilon} \eta_\epsilon(y)k\gamma_{K^\circ}(y)\,dy \\ &\le kA\epsilon \int_{|y|\le\epsilon} \eta_\epsilon(y)\,dy \; = kA\epsilon. \end{aligned}$$

Similarly $|\eta_\epsilon \star \phi - \phi| \le kA\epsilon$.

We can easily show that $\gamma_K(D\phi_\epsilon) \le k$ and $\gamma_K(D\psi_\epsilon) \le k$. Because of Jensen's inequality and convexity of γ_K, we have

$$\begin{aligned} \gamma_K(D\phi_\epsilon(x)) \quad &\le \int \gamma_K(\eta_\epsilon(y)D\phi(x-y))\,dy \\ &= \int \eta_\epsilon(y)\gamma_K(D\phi(x-y))\,dy \\ &\le k\int \eta_\epsilon(y)\,dy \; = k. \end{aligned}$$

Let $U_\epsilon := \{x \in U \mid \phi_\epsilon(x) < \psi_\epsilon(x)\}$, and denote by u_ϵ the minimizer of J_η over $\{v \in H^1(D_\epsilon) \mid \phi_\epsilon \le v \le \psi_\epsilon \text{ a.e. }\}$. Set

$$\begin{aligned} N_\epsilon &:= \{x \in U_\epsilon \mid \phi_\epsilon(x) < u_\epsilon(x) < \psi_\epsilon(x)\} \\ \Lambda_1 &:= \{x \in U_\epsilon \mid u_\epsilon(x) = \phi_\epsilon(x)\} \\ \Lambda_2 &:= \{x \in U_\epsilon \mid u_\epsilon(x) = \psi_\epsilon(x)\}. \end{aligned} \tag{4.8}$$

Since $\phi_\epsilon, \psi_\epsilon$ are smooth, $u_\epsilon \in W^{2,p}(U_\epsilon)$ for any $1 < p < \infty$. Therefore N_ϵ is open and Λ_i's are closed. Also we define the free boundaries $F_i := \partial\Lambda_i \cap U_\epsilon$. Note that ∂N_ϵ consists of F_i's and part of ∂U_ϵ.

Our strategy for the proof is to show that u_ϵ satisfies the bound (4.6) on U_ϵ. Then we can let $\epsilon \to 0$. Since $\phi_\epsilon \to \phi$, $\psi_\epsilon \to \psi$ uniformly, we have $u_\epsilon \to u$ uniformly. Also as for small enough ϵ, u_ϵ's are bounded in $W^{2,\infty}(V)$ for $V \subset\subset U$,

a subsequence of them is weakly star convergent, and the limit is u. Therefore $u \in W^{2,\infty}_{\text{loc}}(U)$ and

$$|D^2 u|_{L^\infty} \le \liminf |D^2 u_\epsilon|_{L^\infty}$$

gives the desired bound.

Now suppose ∂U is not smooth. We approximate U by a shrinking sequence U_i of larger domains with smooth boundaries. Let u_i be the minimizer of J_η on U_i, then $u_i \to u$ uniformly. To see this note that we can consider u as a function on U_i, thus $J_\eta(u_i) \le J_\eta(u)$. An argument similar to the above implies that a subsequence of u_i's converges weakly star to a function $u^\star$, and $u^\star$ satisfies the desired bound. But $u^\star \in W_K$. Also the lower semicontinuity of J_η implies that $J_\eta(u^\star) \le J_\eta(u)$. Since the other inequality is satisfied too, we have $J_\eta(u^\star) = J_\eta(u)$. The uniqueness of the minimizer implies that $u^\star = u$. Hence u satisfies the bound (4.6) too. □

Now let us start proving the bound (4.6) for u_ϵ.

Lemma 7. *We have*

$$\gamma_K(Du_\epsilon) \le k$$

on U_ϵ.

Proof. Since on ∂U_ϵ we have $u_\epsilon = \phi_\epsilon = \psi_\epsilon$ we get $D_z u_\epsilon = D_z\phi_\epsilon = D_z\psi_\epsilon$ for any direction z tangent to ∂U_ϵ, and as u_ϵ is between the obstacles inside U_ϵ we have $D_\nu\phi_\epsilon \le D_\nu u_\epsilon \le D_\nu\psi_\epsilon$ where ν is the normal direction to ∂U_ϵ. Therefore Du_ϵ is a convex combination of $D\phi_\epsilon, D\psi_\epsilon$ and we get the bound on ∂U_ϵ by convexity of γ_K. The bound holds on Λ_i's (and hence on F_i's) obviously as u_ϵ equals one of the obstacles there.

To obtain the bound for N_ϵ note that for any vector z with $\gamma_{K^\circ}(z) = 1$ we have

$$|D_z u_\epsilon| = |z \cdot Du_\epsilon| \le \gamma_{K^\circ}(z)\gamma_K(Du_\epsilon) \le k$$

on ∂N_ϵ, and as $D_z u_\epsilon$ is harmonic in N_ϵ we get $|D_z u_\epsilon| \le k$ in N_ϵ by maximum principle. The result follows from $\gamma_K(Du_\epsilon) = \sup\limits_{\gamma_{K^\circ}(z)=1} |D_z u_\epsilon|$. □

The local behavior of the free boundaries is the same as the case of one obstacle problem as obstacles do not touch inside U_ϵ. We need the following lemma from Friedman [5].

Lemma 8. *The free boundary has measure zero. Furthermore for any direction z*

(i) *if $y \in N_\epsilon$ approaches $x \in F_1$, then $\liminf\limits_{y\to x} D^2_{zz}(u_\epsilon - \phi_\epsilon)(y) \ge 0$.*

(ii) *If $y \in N_\epsilon$ approaches $x \in F_2$, then $\liminf\limits_{y\to x} D^2_{zz}(\psi_\epsilon - u_\epsilon)(y) \ge 0$.*

Lemma 9. *For any direction z with $|z| = 1$, we have*

$$D^2_{zz}\phi_\epsilon(x) \ge -\tfrac{kA^2B}{d_{K^\circ}(x,\partial U) - A\epsilon} \tag{4.9}$$
$$D^2_{zz}\psi_\epsilon(x) \le \tfrac{kA^2B}{d_{K^\circ}(x,\partial U) - A\epsilon}$$

for all $x \in U$ with $d_{K^\circ}(x, \partial U) > A\epsilon$.

Proof. First we assume $\gamma_{K^\circ}(z) = 1$. Let $x_0 \in U$ then

$$\psi(x_0) = c + kd_{K^\circ}(x_0, \partial U) = c + k\gamma_{K^\circ}(x_0 - y_0)$$

for some $y_0 \in \partial U$. Set $\gamma(x) = c + k\gamma_{K^\circ}(x - y_0)$. Then $\psi(x) \le \gamma(x)$ and $\psi(x_0) = \gamma(x_0)$. Now for $h < \gamma_{K^\circ}(x_0 - y_0)$ we have

$$\Delta^2_{h,z}\psi(x_0) = \frac{\psi(x_0 + hz) + \psi(x_0 - hz) - 2\psi(x_0)}{h^2} \le \Delta^2_{h,z}\gamma(x_0). \tag{4.10}$$

By our assumption

$$\Delta^2_{h,z}\gamma(x_0) \le \frac{kB}{\gamma_{K^\circ}(x_0 - y_0) - h} = \frac{kB}{d_{K^\circ}(x_0, \partial U) - h}.$$

Hence $\Delta^2_{h,z}\psi(x) \le \frac{kB}{d_{K^\circ}(x,\partial U)-h}$ for $d_{K^\circ}(x, \partial U) > h$.

Now for $d_{K^\circ}(x, \partial U) > h + A\epsilon$, we have

$$\begin{aligned}
\Delta^2_{h,z}\psi_\epsilon(x) &= \int_{|y|<\epsilon} \eta_\epsilon(y)\Delta^2_{h,z}\psi(x - y)\, dy \\
&\le \int_{|y|<\epsilon} \eta_\epsilon(y)\frac{kB}{d_{K^\circ}(x - y, \partial U) - h}\, dy \\
&\le \int_{|y|<\epsilon} \eta_\epsilon(y)\frac{kB}{d_{K^\circ}(x, \partial U) - A\epsilon - h}\, dy \\
&= \frac{kB}{d_{K^\circ}(x, \partial U) - A\epsilon - h}.
\end{aligned}$$

Here we used the fact that

$$\begin{aligned}
d_{K^\circ}(x - y, \partial U) &\ge d_{K^\circ}(x, \partial U) - \gamma_{K^\circ}(y) \\
&\ge d_{K^\circ}(x, \partial U) - A|y| \\
&> d_{K^\circ}(x, \partial U) - A\epsilon \ > h.
\end{aligned}$$

Taking $h \to 0$, we get for $d_{K^\circ}(x, \partial U) > A\epsilon$

$$D^2_{zz}\psi_\epsilon(x) \le \frac{kB}{d_{K^\circ}(x, \partial U) - A\epsilon}.$$

Now if we take $|z| = 1$ and apply the above result to $w = \frac{z}{\gamma_{K^\circ}(z)}$, we get

$$D^2_{zz}\psi_\epsilon(x) = (\gamma_{K^\circ}(z))^2 D^2_{ww}\psi_\epsilon(x) \le A^2 D^2_{ww}\psi_\epsilon(x) \le \frac{kA^2B}{d_{K^\circ}(x, \partial U) - A\epsilon},$$

as $\gamma_{K^\circ}(z) \le A$ and $D^2\psi_\epsilon$ is nonnegative since ψ is convex. The inequality for ϕ_ϵ follows from $D^2\phi_\epsilon = -D^2\psi_\epsilon$. $\square$

Lemma 10. *For any direction z with $|z| = 1$*

$$|D^2_{zz}u_\epsilon(x)| \le C(n)\big[|\eta| + \frac{kA^2B}{d_{K^\circ}(x, \partial U) - A\epsilon} + \frac{A^2|c|}{(d_{K^\circ}(x, \partial U) - A\epsilon)^2}\big] \tag{4.11}$$

for a.e. $x \in U_\epsilon$, where $C(n)$ is a constant depending only on the dimension n.

Proof. Since $u_\epsilon \in W^{2,p}(U_\epsilon)$ we have $D^2_{zz}u_\epsilon = D^2_{zz}\phi_\epsilon$ a.e. on Λ_1. Also in a $U_\epsilon-$neighborhood of Λ_1 we have $-\Delta u_\epsilon \ge \eta$ a.e., since u_ϵ solves the variational

inequality there. Thus for a.e. $x \in \Lambda_1$

$$\begin{aligned}
-\frac{kA^2B}{d_{K^\circ}(x,\partial U)-A\epsilon} &\le D^2_{zz}\phi_\epsilon(x)\\
&= D^2_{zz}u_\epsilon(x)\\
&= \Delta u_\epsilon(x) - \sum D^2_{z_iz_i}u_\epsilon(x)\\
&\le -\eta - \sum D^2_{z_iz_i}\phi_\epsilon(x)\\
&\le |\eta| + \frac{(n-1)kA^2B}{d_{K^\circ}(x,\partial U)-A\epsilon}, \qquad (4.12)
\end{aligned}$$

where $\{z, z_i\}$ form an orthonormal system (Note that for $x \in U_\epsilon$ we have $d_{K^\circ}(x,\partial U) > A\epsilon$). Similarly, using ψ_ϵ we obtain that for a.e. $x \in \Lambda_2$

$$-|\eta| - \frac{(n-1)kA^2B}{d_{K^\circ}(x,\partial U)-A\epsilon} \le D^2_{zz}u_\epsilon(x) \le \frac{kA^2B}{d_{K^\circ}(x,\partial U)-A\epsilon}. \qquad (4.13)$$

It only remains to obtain the bound on N_ϵ. We do this using maximum principle, since $D^2_{zz}u_\epsilon$ is harmonic in N_ϵ. Therefore we need to estimate $D^2_{zz}u_\epsilon$ near F_i and ∂U_ϵ. First, for $x \in F_1$ and $y \in N_\epsilon$, we have by continuity of $D^2\phi_\epsilon$

$$\liminf_{y\to x} D^2_{zz}u_\epsilon(y) \ge \liminf_{y\to x} D^2_{zz}\phi_\epsilon(y) = D^2_{zz}\phi_\epsilon(x) \ge -\frac{kA^2B}{d_{K^\circ}(x,\partial U)-A\epsilon}. \qquad (4.14)$$

This is true for the z_i directions too. Also

$$\limsup_{y\to x}(D^2_{zz}u_\epsilon(y) + \sum D^2_{z_iz_i}u_\epsilon(y)) = \limsup_{y\to x}\Delta u_\epsilon(y) = -\eta. \qquad (4.15)$$

Thus

$$\begin{aligned}
\limsup_{y\to x} D^2_{zz}u_\epsilon(y) &\le \limsup_{y\to x}(D^2_{zz}u_\epsilon(y) + \sum D^2_{z_iz_i}u_\epsilon(y)) - \sum \liminf_{y\to x} D^2_{z_iz_i}u_\epsilon(y)\\
&= -\eta - \sum \liminf_{y\to x} D^2_{z_iz_i}u_\epsilon(y) \qquad (4.16)\\
&\le |\eta| + \frac{(n-1)kA^2B}{d_{K^\circ}(x,\partial U)-A\epsilon}. \qquad (4.17)
\end{aligned}$$

Similarly on F_2 we have

$$\begin{aligned}
-|\eta| - \frac{(n-1)kA^2B}{d_{K^\circ}(x,\partial U)-A\epsilon} &\le \liminf_{y\to x} D^2_{zz}u_\epsilon(y)\\
\le \limsup_{y\to x} D^2_{zz}u_\epsilon(y) &\le \frac{kA^2B}{d_{K^\circ}(x,\partial U)-A\epsilon}. \qquad (4.18)
\end{aligned}$$

Next we show that

$$|D^2_{zz}u_\epsilon(x)| \le C(n)[|\eta| + \frac{kAB}{r} + \frac{|c|}{r^2}] \qquad x \in N_\epsilon\ d_{K^\circ}(x,\partial U_\epsilon) = Ar, \qquad (4.19)$$

for fixed and small r and $\epsilon < r/16$. Note that

$$d_{K^\circ}(B_{r/2}(x),\partial U_\epsilon) > Ar - Ar/2 = Ar/2.$$

Fix $x_0 \in N_\epsilon$ with $d_{K^\circ}(x_0, \partial U_\epsilon) = Ar$ and consider the function $v_\epsilon(y) = u_\epsilon(x_0 + ry)$ in $B_1(0)$. Then by known bounds on u_ϵ we have in $B_{1/2}(0)$

$$
\begin{aligned}
&|v_\epsilon| \le |c| + 6Ak\epsilon + 3Akr/2 < |c| + 2Akr \\
&\gamma_K(Dv_\epsilon) \le rk.
\end{aligned} \tag{4.20}
$$

Also for a.e. $y \in B_{1/2}(0)$ we have

$$
\begin{aligned}
|\Delta v_\epsilon(y)| \quad &\le nr^2(|\eta| + \frac{(n-1)kA^2B}{d_{K^\circ}(x_0+ry, \partial U) - A\epsilon}) \\
&\le n(|\eta| + \frac{(n-1)kA^2B}{Ar/2 - A\epsilon})r^2 \\
&< n(|\eta| + \frac{16(n-1)kAB}{7r})r^2.
\end{aligned} \tag{4.21}
$$

Since $\Delta u_\epsilon = -\eta$ in N_ϵ and it is bounded on Λ_i's (and free boundaries have measure zero). Choose $\sigma \in C_0^\infty(B_{1/2}(0))$ such that $\sigma = 1$ in $B_{1/4}(0)$. Then in $B_{1/2}(0)$

$$
\begin{aligned}
|\Delta(\sigma v_\epsilon)| \quad &= |(\Delta\sigma)v_\epsilon + 2D\sigma \cdot Dv_\epsilon + \sigma\Delta v_\epsilon| \\
&\le C(n)[(|\eta| + \frac{kAB}{r})r^2 + |c|].
\end{aligned} \tag{4.22}
$$

By elliptic theory it follows

$$
|\sigma v_\epsilon|_{W^{2,p}(B_{1/2}(0))} \le C(n,p)[(|\eta| + \frac{kAB}{r})r^2 + |c|], \tag{4.23}
$$

for any $1 < p < \infty$ (note that boundary term is zero). In particular

$$
|D^2_{ij} v_\epsilon|_{L^p(B_{1/4}(0))} \le C(n,p)[(|\eta| + \frac{kAB}{r})r^2 + |c|]. \tag{4.24}
$$

We want to extend this to $p = \infty$.

Let $\tau \in C_0^\infty(B_{1/4}(0))$ with $\tau = 1$ in $B_{1/8}(0)$. Consider the open set $N = \{y \mid x_0 + ry \in N_\epsilon\}$. In N we have $\Delta v_\epsilon = -\eta r^2$. Thus (note that v_ϵ is smooth in N)

$$
\Delta D^2_{zz}(\tau v_\epsilon) = D_z h, \tag{4.25}
$$

where

$$
\begin{aligned}
h \quad := D_z\Delta(\tau v_\epsilon) \quad &= D_z((\Delta\sigma)v_\epsilon + 2D\sigma \cdot Dv_\epsilon + \sigma\Delta v_\epsilon) \\
&= D_z((\Delta\sigma)v_\epsilon + 2D\sigma \cdot Dv_\epsilon - \sigma\eta r^2).
\end{aligned} \tag{4.26}
$$

Using the above estimates we find that

$$
|h|_{L^p(N)} \le C(n,p)[(|\eta| + \frac{kAB}{r})r^2 + |c|]. \tag{4.27}
$$

Now take

$$
V(y) = \begin{cases} \alpha_n |y|^{2-n} & n \ge 3 \\ \\ \alpha_2 \log|y| & n = 2 \end{cases} \tag{4.28}
$$

to be the fundamental solution of $-\Delta$. Then

$$
g(y) = -\int_N \frac{\partial V(y-w)}{\partial z} h(w)\, dw
$$

satisfies

$$\Delta g = \frac{\partial h}{\partial z}$$

in N. By the bound on h we find that

$$|g|_{L^\infty(N)} \leq C(n)[(|\eta| + \frac{kAB}{r})r^2 + |c|], \tag{4.29}$$

since for $p > n$, $\frac{\partial V}{\partial z}$ is in L^q where q is the dual exponent of p. The function $D^2_{zz}(\tau v_\epsilon) - g$ is then harmonic in $N \cap B_{1/4}(0)$. The boundary of this set consists of part of $\partial B_{1/4}(0)$ in which $\tau = 0$ and g is bounded, and another part inside $B_{1/4}(0)$ where corresponds to the free boundaries and both g, $D^2_{zz}(\tau v_\epsilon)$ are bounded there by the above bounds. Therefore by the maximum principle we get

$$|D^2_{zz} v_\epsilon(0)| \leq C(n)[(|\eta| + \frac{kAB}{r})r^2 + |c|]. \tag{4.30}$$

Hence

$$\begin{aligned} |D^2_{zz} u_\epsilon(x_0)| &\leq C(n)[|\eta| + \frac{kAB}{r} + \frac{|c|}{r^2}] \\ &= C(n)[|\eta| + \frac{kA^2B}{d_{K^\circ}(x_0, \partial U_\epsilon)} + \frac{A^2|c|}{(d_{K^\circ}(x_0, \partial U_\epsilon))^2}]. \end{aligned}$$

The proof of the lemma is complete once we notice that for $x \in U_\epsilon$

$$d_{K^\circ}(x, \partial U_\epsilon) \geq d_{K^\circ}(x, \partial U) - A\epsilon.$$

□

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Department of Mathematics, UC Berkeley, Berkeley, CA 94720, USA
safdari@berkeley.edu